Some Prismatic Dieudonne Crystals

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These are lecture notes for a talk on prismatic Dieudonne theory. They haven't been proofread, so be careful.

1 Introduction:

The goal of these notes is two-fold: first, we want relatively concrete descriptions of the prismatic Dieudonne crystals associated to etale p-divisible groups and groups of multiplicative type. The second goal is to begin the proof that the prismatic Dieudonne functor

$$M_{\Delta} : BT(R) \to DF(R)$$

is fully faithful.

Definition 1.1. *A p-divisible group $G = (G_n, i_n)$ is said to be* **etale** *if each G_n is finite etale. It is said to be of* **multiplicative type** *if locally it's isomorphic to μ_p^{∞}.*

1.1 Main Results:

In order to state the main results of this section, it's useful to describe the prismatic Dieudonne crystal in a slightly different way, in terms of the p-adic Tate module.

Lemma 1.1. *For any p-divisible group G over R, there's a canonical isomorphism*

$$\mathcal{M}_{\Delta}(G) \simeq \mathcal{H}om_{(R)_{qsyn}}(T_p(G), \mathcal{O}^{pris}).$$

The description of the Dieudonne crystals mimics almost exactly the analogous description in the crystalline case. For the etale case, the results and arguments really are identical.

Theorem 1.2. *Let G be an etale p-divisible group. Then the prismatic Dieudonne crystal is given by*

$$\mathcal{M}_{\Delta}(G) \simeq \mathcal{H}om(T_p(G), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}^{pris}.$$

In particular, $\mathcal{M}_{\Delta}(\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathcal{O}^{pris}$.

It turns out we won't be able to describe $\mathcal{M}_\Delta(\mu_{p^\infty})$ in it's full glory, but will obtain a nice description upon restricting $\mathcal{M}_\Delta(\mu_{p^\infty})$ to the cyclotomic base prism $(\mathbb{Z}_p[[q-1]], [p]_q)$. After this restriction, the crystal is freely generated by a single element - a q-deformation of the logarithm. Define $\mathbb{Z}_p(1) := T_p(\mu_{p^\infty})$. Then the q-logarithm defines a map of sheaves

$$l_q : u^{-1}(\mathbb{Z}_p(1)) \to \mathcal{O}_\Delta$$

(this map only makes sense over the cyclotomic prism - we'll discuss this in detail later) which yields an element of $\mathcal{M}_\Delta(\mu_{p^\infty})$. Our description of $\mathcal{M}_\Delta(\mu_{p^\infty})$ then takes the following form

Theorem 1.3 (Theorem 3.6)**.** *Over $(\mathbb{Z}_p[[q-1]], [p]_q)$, the prismatic crystal*

$$\mathcal{H}om_{(R)_\Delta}(u^{-1}(\mathbb{Z}_p(1)), \mathcal{O}_\Delta)$$

is freely generated by l_q, and the Frobenius acts on this element via

$$l_q \to [p]_q l_q.$$

With these descriptions in hand, the action of M_Δ on maps from $\mathbb{Q}_p/\mathbb{Z}_p$ to μ_{p^∞} admits a simple description in terms of the q logarithm. We can utilize this description to prove the main result of this lecture:

Theorem 1.4. *The prismatic Dieudonne functor induces a bijection*

$$Hom_{BT(R)}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \xrightarrow{M_\Delta} Hom_{DF(R)}(M_\Delta(\mu_p^\infty), M_\Delta(\mathbb{Q}_p/\mathbb{Z}_p)).$$

The proof relies on an explicit computation of the cyclotomic trace, thus importing some rather heavy machinery from algebraic K-theory. I intend to devote the second half of the lecture to unpacking this computation.

While the main theorem of todays lecture may seem to be a far cry from the desired fully-fatihfulness of the Dieudonne functor, it turns out that by some witch-craft (inspired by deJong and Scholze-Weinstein), one can reduce the proof of fully-faithfulness to this single computation! Kirill will be our guide to this argument next week.

2 Etale p-divisible groups and their crystals:

Let's begin by proving our promised description of the Dieudonne crystal of a general p-divisible group in terms of the Tate module. The idea is that for any p-divisible group G, we can pass to a 'universal cover' $\tilde{G}$ which is a $\mathbb{Q}_p$-vector space. Namely, we define $\tilde{G}$ as the inverse limit

$$\tilde{G} := \lim(... \xrightarrow{\times p} G \xrightarrow{\times p} G)$$

By construction, this is a $\mathbb{Q}_p$-vector space, and unsurprisingly we have the following:

Lemma 2.1. *There's a short exact sequence*

$$0 \to T_pG \to \tilde{G} \to G \to 0$$

in $Ab(R_{qsyn})$*.*

The proof of this lemma relies on 'repleteness' of the topos $Shv(R_{qsyn})$ in order to pass from the obvious short exact sequences

$$0 \to G[p^n] \to G \to G \to 0$$

to the limit. I omit the argument. With this in hand, we obtain the following:

Lemma 2.2. *For any p-divisible group G, there's a canonical isomorphism*

$$\mathcal{M}_\Delta(G) \simeq \mathcal{H}om(T_p(G), \mathcal{O}^{pris}).$$

Proof. Since $\tilde{G}$ is a sheaf of $\mathbb{Q}_p$-vector spaces and $\mathcal{O}^{pris}$ is p-complete, the entire derived functor $R\mathcal{H}om(\tilde{G}, \mathcal{O}^{pris})$ vanishes, so the long exact sequence induced from the short exact sequence above yields the desired isomorphism. □

Now there's a canonical map of sheaves $\mathbb{Z}_p \to \mathcal{O}^{pris}$, which induces a map

$$\mathcal{H}om(T_p(G), \mathbb{Z}_p) \to \mathcal{H}om(T_p(G), \mathcal{O}^{pris}) \simeq \mathcal{M}_\Delta(G)$$

of $\mathbb{Z}_p$-modules. By adjunction, this induces a map of $\mathcal{O}^{pris}$-modules

$$u_G : \mathcal{H}om(T_p(G), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}^{pris} \to \mathcal{M}_\Delta(G)$$

and the point is that when G is etale, this map is an isomorphism.

Theorem 2.3. *If G is an etale p-divisible group, the map u_G is an isomorphism. In particular,* $\mathcal{M}_\Delta(\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathcal{O}^{pris}$*.*

Proof. Recall, G is etale if and only if each G_n (in the defining inverse system) is finite etale. Whence, each G_n defines a locally free $\mathbb{Z}/p^n\mathbb{Z}$-module on $(R)_{et}$, and by taking the Tate module $T_p(G) := \lim G[p^n] = \lim G_n$, we see that $T_p(G)$ is an etale-locally free $\mathbb{Z}_p$-module. Since both the domain and codomain of u_G are, in particular, etale sheaves, we can localize in the etale topology. This reduces us to showing that the canonical map

$$\mathcal{H}om(\mathbb{Z}_p^n, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}^{pris} \to (\mathcal{O}^{pris})^n$$

is an isomorphism, where n is the height of G. This is clear. □

With this lemma in hand, we get a nice description of the action of M_Δ on maps from $\mathbb{Q}_p/\mathbb{Z}_p$ to any other p-divisible group G. Recall, the Tate module of G can be described as $T_p(G) \simeq \mathcal{H}om_{(R)_{qsyn}}(\mathbb{Q}_p/\mathbb{Z}_p, G)$, so applying M_Δ to this hom-set, and using the identification of $M_\Delta(\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathcal{O}^{pris}$, we get a map

$$T_p(G)(R) \simeq Hom_{(R)_{qsyn}}(\mathbb{Q}_p/\mathbb{Z}_p, G) \xrightarrow{\mathcal{M}_\Delta} Hom_{D\mathcal{M}(R)}(\mathcal{H}om(T_p(G), \mathcal{O}^{pris}), \mathcal{O}^{pris})$$

and this behaves exactly as you'd guess: namely, for $x \in T_p(G)$, $M_\Delta(x)$ is given by evaluating a given morphism $f : T_p(G) \to \mathcal{O}^{pris}$ at x.

3 Multiplicative Groups and Their Crystals

3.1 Logarithms:

Recall, if k is a quasi-syntomic ring of characteristic p, the crystalline site $(k/\mathbb{Z}_p)_{CRIS}$ has as objects triples (T, U, δ), where T is some scheme over k, $U \to T$ is a closed immersion, defined by some ideal I_T, and δ is the data of a divided power structure on this ideal. The structure sheaf $\mathcal{O}^{crys}$ is defined by

$$\mathcal{O}^{crys}((T, U, \delta)) = \mathcal{O}_T(T)$$

and the ideal sheaf $\mathcal{I}^{crys}$ is defined by

$$\mathcal{I}^{crys}((T, U, \delta)) = Ker(\mathcal{O}_T(T) \to \mathcal{O}_U(U))$$

Let's focus on the affine setting for a moment, so such a triple is a surjection of k-algebras $R' \to R$ and a PD ideal I. Since R' is (p, I)-adically complete, we can associate to the surjection a Teichmuller lift

$$[-] : R^\flat := \lim_{x \to x^p} R \to R'$$

given by sending $(x_1, x_2, ...) \to \lim_n \tilde{x}_n^{p^n}$ where $\tilde{x}_n$ is just *any lift* of x_n to R'. It turns out this doesn't depend on the choice of lift (thanks to (p, I)-completeness), and the composite

$$R^\flat \xrightarrow{[-]} R' \to R$$

just projects onto the first coordinate, essentially by definition.

If we're in the business of studying multiplicative p-divisible groups, we're interested in μ_{p^∞}. Define $\mathbb{Z}_p(1) := T_p(\mu_{p^\infty})$ and notice that $\mathbb{Z}_p(1)(R) \simeq T_p(R^\times)$ includes into $R^\flat$ as the elements whose first coordinate is equal to 1! So applying the Teichmuller lift, we land in $1 + \mathcal{I}$, and by the existence of divided powers on $\mathcal{I}$, we can define the crystalline logarithm $log : 1 + \mathcal{I} \to \mathcal{O}^{crys}$ by

$$log(x) = \sum_{n=1}^{\infty} (-1)^n (n-1)! \gamma_n (x - 1).$$

Combining these observations, we produce a map of sheaves $log : \mathbb{Z}_p(1) \to \mathcal{O}^{crys}$, thus yielding an element of the crystalline Dieudonne crystal

$$\mathbb{D}(\mu_{p^\infty}) = \mathcal{E}xt^1(i_\star^{crys}\mu_{p^\infty}, \mathcal{O}^{crys}) \simeq \mathcal{H}om(i_\star^{crys}\mathbb{Z}_p(1), \mathcal{O}^{crys}).$$

The fundamental fact here is the following theorem of Berthelot-Green-Messing:

Theorem 3.1. *The logarithm freely generates the Dieudonne crystal* $\mathbb{D}(\mu_{p^\infty})$.

Unfortunately, when we enter the realm of quasi-syntomic rings, the logarithm is insufficient for our purposes. If we were going to mimic the above story in the prismatic setting, we'd need to produce some kind of 'logarithm'

$$T_p(R^\times) \to \Delta_R,$$

but we have little hope of making sense of this in general. Of course, we know we can restrict to quasi-regular semi-perfectoids, and at least in this setting Δ_R is an actual ring (and not just some object in a derived category), and we can actually make sense of the Teichmuller lift: The canonical map $R \to \overline{\Delta}_R$ induces a map

$$[-]_\theta : R^\flat \to \overline{\Delta}_R^\flat \xrightarrow{[-]} \Delta_R.$$

But now, if we try to carry out the above procedure we run into the following obstruction: the crystalline logarithm made sense because *we were handed a divided power structure* as part of the data of the crystalline site! Of course, if we restrict to characteristic p, we can recover this data via the crystalline comparison, but then we might as well just work in the crystalline setting from the get-go.

The solution to this conundrum is to work with $\mathbb{Z}_p^{cyc}$-algebras. Recall, $\mathbb{Z}_p^{cyc}$ is obtained by adjoining a compatible system of p^n-th roots of unity to $\mathbb{Z}_p$, and then completing

$$\mathbb{Z}_p^{cyc} := (colim_n \mathbb{Z}_p[\zeta_{p^n}])^\wedge$$

This is a perfectoid ring, and the associated perfect prism $(A_{inf}(\mathbb{Z}_p^{cyc}), I)$ is the perfection of the cyclotomic prism $(\mathbb{Z}_p[[q-1]], [p]_q)$. Here's a fundamental fact about perfectoid rings, proven in Bhatt and Scholze section 7.

Lemma 3.2 (Andre's Lemma)**.** *Given a perfectoid ring R, there's a quasi-syntomic cover $R \to S$ where S is absolutely integrally closed - i.e. every monic polynomial with coefficients in S has a solution in S.*

By Andre's lemma given any quasi-syntomic ring, we can pass to a quasi-syntomic cover over $\mathbb{Z}_p^{cyc}$ (we just need to add a whole bunch of roots of unity), and since everything we're working with is a stack on the quasi-syntomic site, this imposes essentially no loss of generality for the types of arguments we can make. On the other hand, as soon as we're working over $\mathbb{Z}_p^{cyc}$, we can produce the divided powers we missed so dearly a moment ago.

Lemma 3.3. *Let $(A, (\xi))$ be a prism over $(\mathbb{Z}_p[[q-1]], [p]_q)$ (so $\xi = [p]_q$) and let $x, y \in A$ be rank 1 elements such that $\varphi(x-y) \in \xi A$. Then the q-deformed divided power*

$$\gamma_{q,n}(x-y) := \frac{(x-y)(x-qy)...(x-q^{n-1}y)}{[n]_q!}$$

exists, and lies in $\mathcal{N}^{\geq n}A$.

So to reiterate the point made above: in our general setting of quasi-syntomic rings, the quasi-regular semi-perfectoid $\mathbb{Z}_p^{cyc}$-algebras occupy a privileged position. They're general enough that we can bootstrap results about these rings up to all quasi-syntomic rings by descent arguments, and yet have enough structure that we can extract some semblance of divided powers, which opens the door to cohomological tools (q-deRham cohomology!) heretofore unavailable to us.

Sketch of proof: Observe the following property of the cyclotomic prism: the sequence $(p, [p]_q)$ is regular (such prisms are called 'transversal'). Any prism (B, I) over $(\mathbb{Z}_p[[q-1]], [p]_q)$ where A is flat over $\mathbb{Z}_p[[q-1]]$ then necessarily satisfies this property, and we can reduce to this case. Indeed, to name two elements x, y as in the statement of the lemma is to name a map

$$\mathbb{Z}_p[[q-1]]\langle x, y, \lambda\rangle / (x^p - y^p - \xi\lambda) \to A$$

and $\mathbb{Z}_p[[q-1]]\langle x, y, \lambda\rangle / (x^p - y^p - \xi\lambda)$ is flat over $\mathbb{Z}_p[[q-1]]$. So we may assume that (A, I) is a transversal prism. The import of this reduction comes from the following facts whose proof we will omit (their proofs can be found in the paper of Anschutz-LeBras on the cyclotomic trace).

1. If $(A, (\xi))$ is a transversal prism, then the sequences $(\varphi^{r-1}(\xi), \xi)$ are regular for all $r \geq 2$.
2. Let n be an integer and for all $r \geq 0$, define integers $a_r, b_r \geq 0$, $b_r < p^r$ such that

   $$n = a_r p^r + b_r.$$

   Then

   $$[n]_q! = u \prod_{r \geq 1} (\varphi^{r-1}(\xi))^{a_r}$$

   for some unit $u \in \mathbb{Z}_p[[q-1]]$.

Appealing to the second fact, we see that it suffices to show that $(x-y)...(x-q^{n-1}y)$ is divisible by $\prod_{r\geq 1} \varphi^{r-1}(\xi)^{a_r}$, and by the first fact, it suffices to show that each factor of this product divides $(x-y)...(x-q^{n-1}y)$. To show this, it suffices to assume $b_r = 0$ - i.e. $n = a_r p^r$.

The idea now is to split the product $(x-y)...(x-q^{n-1}y)$ up in to a_r distinct factors, and show each factor is divisible by $\varphi^{r-1}(\xi)$. Namely, for $0 \le k < a_r$, define

$$z_k = (x - q^{kp^r} y)(x - q^{kp^r+1} y)...(x - q^{(k+1)p^r-1} y).$$

Now observe that, as a polynomial in q, $\varphi^{r-1}([p]_q)$ is the p^r-th cyclotomic polynomial. So in particular, we have the congruence

$$x^{p^r} - y^{p^r} = (x-y)(x-qy)...(x-q^{p^r-1}y) \bmod \varphi^{r-1}([p]_q).$$

Substituting $q^{kp^r}y$ in place of y in the above congruence, we obtain congruences

$$z_k = x^{p^r} - q^{l_k} y^{p^r} \bmod \varphi^{r-1}(\xi)$$

where $l_k = \lambda_k p^r$ for some λ_k. So we are reduced to showing that $x^{p^r} - q^{l_k} y^{p^r}$ is divisible by $\varphi^{r-1}(\xi)$.

Write

$$x^{p^r} - q^{l_k} y^{p^r} = x^{p^r} - y^{p^r} - (q^{l_k} - 1) y^{p^r}$$

Now by our hypothesis that x and y are rank 1 and $\varphi(x-y) \in (\xi)$, we see

$$x^{p^r} - y^{p^r} = \varphi^r(x-y) = \varphi^{r-1}(x^p - y^p)$$

is divisible by $\varphi^{r-1}(\xi)$. On the other hand,

$$q^{l_k} - 1 = \varphi^{r-1}(\xi) \frac{q^{l_k} - 1}{q^{p^r} - 1} (q-1)$$

since $l_k = \lambda_k p^r$. Thus z_k is divisible by $\varphi^{r-1}(\xi)$, as was to be shown.

All that remains is the statement about the Nygaard filtration. We know that $\varphi(x-qy) \in (\xi)$ by hypothesis, so it suffices to show $\varphi([n]_q!)$ is not divisible by ξ. But again appealing to fact number 2 above, we see that

$$\varphi([n]_q!) = u \prod_{r \ge 1} \varphi^r(\xi)$$

and the $\varphi^r(\xi)$ are not divisible by ξ by fact 1, so we win.

$\square$

Notice, if we base change to $\mathbb{Z}_p$, in effect setting $q = 1$, then $[n]_q = n$ and $\gamma_{n,q} = \dfrac{(x-y)^n}{n!}$ is the usual divided power element on $\mathbb{Z}_p$. So working with quasi-regular semi-perfectoid $\mathbb{Z}_p^{cyc}$-algebras effectively lifts the divided power structures from the characteristic p case to our new setting. Furthermore, we can now build our logarithm!

Lemma 3.4. *Let (A, I) be a prism over $(\mathbb{Z}_p[[q-1]], [p]_q)$ and let $x \in 1 + \mathcal{N}^{\geq 1}A$ be a rank 1 element. Then the q-logarithm*

$$log_q(x) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n(n-1)}{2}} \frac{(x-1)...(x-q^{n-1})}{[n]_q}$$

is well defined and converges. Furthermore, modulo $\mathcal{N}^{\geq 2}A$, the q-logarithm assumes the form

$$log_q(x) = x - 1 \ mod \mathcal{N}^{\geq 2}A.$$

Proof. We simply appeal to the existence of q-divided powers to rewrite the sum as

$$log_q(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n(n-1)}{2}} [n-1]_q! \gamma_{n,q}(x-1).$$

Now by $[p]_q$-adic completeness (our prism lies over the cyclotomic prism!) we see that $[n-1]_q! \to 0$ as $n \to \infty$, so the sum converges.

□

Now we're in a position to attempt to replicate the successes of the crystalline story in our more general setting. The first step is to use the q-logarithm to define a map of sheaves

$$l_q : u^{-1}(\mathbb{Z}_p(1)) \to \mathcal{O}_{\Delta},$$

on $(\mathbb{Z}_p^{cyc})_{\Delta}$. We just need to verify the hypotheses on the ranks of the specific elements.

Construction: Let R be a quasi-regular semi-perfectoid $\mathbb{Z}_p^{cyc}$-algebra. For any $(A, I) \in (R)_{\Delta}$, we identify $u^{-1}(\mathbb{Z}_p(1))(A, I) \simeq T_p((A/I)^{\times})$. Now the surjection $A \to A/I$ yields a Teichmuller lift

$$(A/I)^{\flat} \xrightarrow{[-]} A$$

and we define the map

$$[-]_{\theta} : u^{-1}(\mathbb{Z}_p(1))(A, I) \to A$$

as the composite

$$T_p((A/I)^{\times}) \to (A/I)^{\flat} \xrightarrow{(-)^{\frac{1}{p}}} (A/I)^{\flat} \xrightarrow{[-]} A.$$

Lemma 3.5. *For any $x \in T_p((A/I)^{\times})$, the element $[x]_{\theta} \in A$ is of rank 1 and lies in $1 + \mathcal{N}^{\geq 1}A$.*

Proof. To see that it's rank 1, simply observe that for any $a \in A$, applying δ to the a^p lies in pA;

$$\delta(a^p) = pa^{p^2-p}\delta(a) + \sum_{i=1}^{p-1} pa^{p^{i-1}}\delta(a)\delta(a^{p-i}).$$

In particular, any element x that admits arbitrarily large p-th roots satisfies

$$\delta(x) \in \bigcap_n (p^n) = 0.$$

Of course, any element in the image of the Teichmuller map admits arbitrarily large p-th roots essentially by definition. So $[x]_\theta$ is of rank 1. For the second claim, just apply Frobenius to see

$$\varphi([x]_\theta) = [x] = 1 \bmod [p]_q A$$

so the claim follows.

□

Of course, one of our main tools is to bootstrap the arguments of Berthelot-Breen-Messing up to the quasi-syntomic case, so we had better check the compatibility of our q-logarithm with the crystalline logarithm in the characteristic p-case. Let R be a quasi-regular semi-perfectoid ring of characteristic p. Recall, the crystalline period ring $A_{crys}(R)$ is the universal p-complete PD-thickening of R. Since p is a non-zero divisor in $A_{crys}(R)$ and this ring is equipped with a canonical Frobenius lift, we see that $(A_{crys}(R), (p))$ is a prism, and thus by the universal property of prismatic cohomology, we get a canonical map

$$\Delta_R \to A_{crys}(R)$$

compatible with Frobenius. This map turns out to be an isomorphism, and carries $1 + \mathcal{N}^{\geq 1}\Delta_R$ to $1 + \mathcal{I}_{crys}(R)$.

In the case where $R = W(R^\flat)/(x)$ for some non-zero divisor, we can compute both the prismatic cohomology, as well as the universal PD-thickening. Both are given by prismatic envelopes:

$$\Delta_R \simeq \Delta_{R/W(R^\flat)} \simeq W(R^\flat)\{\frac{x}{p}\}^\wedge$$

and

$$A_{crys}(R) \simeq W(R^\flat)\{\frac{x^p}{p}\}^\wedge \simeq \Delta_R \otimes_{W(R^\flat),\varphi} W(R^\flat).$$

So for example, if $R = k$ is a perfect field of characteristic p, then the isomorphism

$$W(k) \to A_{crys}(k) \simeq W(k)$$

is given by the Frobenius. In particular, we see that the diagram of Teichmuller lifts

$$\begin{array}{ccc} & & 1+\mathcal{N}^{\geq 1}\Delta_k \\ & \nearrow & \downarrow \\ T_p(k^\times) & & \\ & \searrow & \\ & & 1+\mathcal{I}_{crys}(k) \end{array}$$

commutes, which tells us that the corresponding logarithms agree as well, when R is a perfect field of characteristic p. This will suffice for our reductions.

3.2 The Dieudonne Crystals of Multiplicative Groups:

So we've succeeded in constructing a logarithm

$$l_q : u^{-1}(\mathbb{Z}_p(1)) \to \mathcal{O}_\Delta$$

for any QRSPerf $\mathbb{Z}_p^{cyc}$-algebra.

Theorem 3.6. *Over $(A, I) = (\mathbb{Z}_p[[q-1]], ([p]_q))$, the prismatic crystal*

$$\mathcal{H}om_{(R)_\Delta}(u^{-1}(\mathbb{Z}_p(1)), \mathcal{O}_\Delta)$$

is free of rank 1, generated by l_q. Moreover, the Frobenius sends

$$l_q \to [p]_q l_q.$$

Observe that since $\hat{\mathbb{G}}_m/\mu_{p^\infty}$ is uniquely p-divisible, we can compute the crystal via

$$\mathcal{H}om_{(R)_\Delta}(u^{-1}(\mathbb{Z}_p(1)), \mathcal{O}_\Delta) \simeq \mathcal{E}xt^1_{(R)_\Delta}(u^{-1}(\hat{\mathbb{G}}_m), \mathcal{O}_\Delta).$$

Proof. To show that this is free, it suffices to see that the global ext groups

$$Ext^1(u^{-1}(\hat{\mathbb{G}}_m)\big|_{(B,J)}, \mathcal{O}_\Delta)$$

are free of rank 1 for all (B, J) over the cyclotomic base (so the Ext is taken in the localized site $(R)_\Delta/(B, J)$), since this shows that all sections of the sheaf are free. At this point, we already know that the sheaf in question is a crystal, and so satisfies base change. Thus, we can even reduce to the case of (A, I).

Now to compute Ext^1, we want to use the Berthelot-Breen-Messing spectral sequence to reduce to a computation of prismatic cohomology. The terms of the spectral sequence are of the form $H^i(\hat{\mathbb{G}}_m^j, \mathcal{O}_\Delta)$ (or products of these)

Furthermore, $\hat{\mathbb{G}}_m \simeq Spf(R\langle x,y\rangle/(xy-1))$, and thus the base change is given by $\hat{\mathbb{G}}_m \times_{Spf(R)} Spf(A/I) \simeq Spf(\mathbb{Z}_p[\zeta_p]\langle x,y\rangle/(xy-1))$. The key input is we now can appeal to a comparison (a theorem of Bhatt and Scholze) with q-crystalline cohomology:

$$\Delta_{\hat{\mathbb{G}}_m/A} \simeq q\Omega_{\tilde{R}/\mathbb{Z}_p[[q-1]]}$$

where $\tilde{R} = \mathbb{Z}_p[[q-1]]\langle x,y\rangle/(xy-1)$ (the omission of the root of unity is not a mistake - this is built into the comparison isomorphism). By choosing etale coordinates

$$\mathbb{Z}_p[[q-1]]\langle x\rangle \xrightarrow{x\to x} \tilde{R}$$

we can compute the q-crystalline cohomology via an extremely explicit complex - the q-deRham complex. This takes the form

$$\mathbb{Z}_p[[q-1]]\langle x,y\rangle/(xy-1) \xrightarrow{\Delta_q} \mathbb{Z}_p[[q-1]]\langle x,y\rangle/(xy-1)d_qx$$

where $\Delta_q(f(x)) = \frac{f(qx)-f(x)}{qx-x}d_qx$.

The zeroth cohomology of this complex is just $\mathbb{Z}_p[[q-1]]$, and thus the bottom row of the BBM spectral sequence is exact by a direct computation, so

$$Ext^1(u^{-1}\hat{\mathbb{G}}_m, \mathcal{O}_\Delta) \simeq Ker(H^1(\hat{\mathbb{G}}_m\mathcal{O}_\Delta) \to H^1(\hat{\mathbb{G}}_m^{\times 2}, \mathcal{O}_\Delta)).$$

The element l_q in the Ext group is sent to the class represented by $\frac{d_qx}{x}$, which generates a free submodule of H^1. So the natural map

$$\mathbb{Z}_p[[q-1]] \xrightarrow{a\to a\cdot l_q} Ext^1(u^{-1}\hat{\mathbb{G}}_m, \mathcal{O}_{pris})$$

is injective.

We know from section 4.6 that the Ext group in question is finitely generated. Thus, to show that in fact it's generated by l_q, we can appeal to Nakayama's lemma to pass to the case of $(B,J) = (W(k),(p))$ where k is an algebraically closed field of characteristic p. Then by the crystalline comparison from section 4.3, the prismatic Dieudonne crystal is isomorphic to the crystalline Dieudonne crystal, and the q-logarithm reduces to the crystalline logarithm - which we know generates the crystalline Dieudonne crystal! So we win.

Finally, the statement about the Frobenius follows by direct computation. Namely, we can express the q-logarithm in terms of the usual logarithm, via

$$log_q(x) = \frac{q-1}{log(q)}log(x)$$

where $log(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x-1}{n}$ (this equality takes place not in B but in $B[1/p][[x-1]]^\wedge$). Applying Frobenius, we obtain

$$\varphi(log_q(x)) = \frac{q^p - 1}{log(q^p)} log(x^p) = \frac{q^p - 1}{q-1}\frac{q-1}{log^q} log(x) = [p]_q log_q(x)$$

as desired.

∎

As a bonus, we obtain a simplified description of the Dieudonne crystals of any p-divisible group of multiplicative type, provided R is a $\mathbb{Z}_p[\zeta_p]$-algebra.

Corollary 3.6.1. *Let G be a multiplicative p-divisible group over R. Then there is a canonical isomorphism*

$$u^{-1}(\mathcal{H}om(G, \mu_{p^\infty})) \otimes_{\mathbb{Z}_p} \mathcal{O}_\Delta \simeq \mathcal{E}xt^1_{(R)_\Delta}(u^{-1}G, \mathcal{O}_\Delta)\big|_{(R/A)_\Delta}$$

given by sending $f : G \to \mu_{p^\infty}$ to the evaluation of the morphism induced by f.

4 Towards Fully Faithfulness:

Our descriptions of $M_\Delta(\mu_{p^\infty})$ and $M_\Delta(\mathbb{Q}_p/\mathbb{Z}_p)$ lend themselves to a simple expression of the hom set between these modules:

Lemma 4.1. *For any quasi-regular semi-perfectoid $\mathbb{Z}_p^{cyc}$-algebra R, there's a canonical identification*

$$Hom_{DM(R)}(M_\Delta(\mu_{p^\infty}), M_\Delta(\mathbb{Q}_p/\mathbb{Z}_p)) \simeq \Delta_R^{\varphi=\hat{\xi}}.$$

Proof. We know $M_\Delta(\mathbb{Q}_p/\mathbb{Z}_p) \simeq \Delta_R$, and since $M_\Delta(\mu_{p^\infty})$ is freely generated by l_q, any between the modules is determined by the image of l_q. Of course, any morphism of Dieudonne modules must respect the Frobenius endomorphism, so the only constraint on the image of l_q is imposed by the action of Frobenius, which we know to act on l_q via multiplication by $\hat{\xi} = [p]_q$. The identification follows. ∎

Finally, at the end of our discussion of etale groups, we gave a description of the action of M_Δ on $Hom(\mathbb{Q}_p/\mathbb{Z}_p, -)$. Combining this description with the previous lemma, we obtain:

Corollary 4.1.1. *The action of M_Δ on $Hom(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$ is identified with the q-logarithm*

$$\begin{array}{ccc} Hom(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) & \xrightarrow{M_\Delta} & Hom_{DM(R)}(M_\Delta(\mu_{p^\infty}), M_\Delta(\mathbb{Q}_p/\mathbb{Z}_p)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbb{Z}_p(1) & \xrightarrow{l_q} & \Delta_R^{\varphi=\hat{\xi}} \end{array}$$

To prove that this map is a bijection, we need to appeal to the topological description of prismatic cohomology. By work of Bhatt-Morrow-Scholze, the Nygaard completed prismatic cohomology of any quasi-regular semi-perfectoid ring can be identified with $\pi_0(TC^-(R;\mathbb{Z}_p))$. This identification preserves the Nygaard filtration on both sides (where for $\pi_0(TC^-)$ the 'Nygaard filtration' is the double-speed abutment filtration of the homotopy fixed point spectral sequence), and is compatible with Frobenius. The higher pieces of the Nygaard filtration can be identified with the higher homotopy groups of $TC^-(R;\mathbb{Z}_p)$, and in particular, the fiber sequence

$$TC \to TC^- \xrightarrow{can-\varphi^{hT}} TP$$

identifies $\pi_2(TC(R;\mathbb{Z}_p)) \simeq \hat{\Delta}_R^{\varphi=\xi}$.

Now these topological theories play the role of approximating algebraic K-theory. In particular, theres a map

$$Ctr : K(R;\mathbb{Z}_p) \to TC(R;\mathbb{Z}_p)$$

called the *cyclotomic trace* that plays a central role in modern algebraic K-theory - indeed, in many cases it can be shown to be an isomorphism!

Now, the p-adic Tate module admits a map to degree 2 p-adic K theory for very elementary reasons, and the amazing fact is that the map we just described on the hom sets of the Dieudonne modules *is the cyclotomic trace in degree 2*. Recall, we defined l_q by the formula

$$l_q(x) = log_q([x^{\frac{1}{p}}]).$$

For simplicity of notation, we'll denote by $-l_q$ the map

$$-l_q(x) = log_q([x^{-\frac{1}{p}}]).$$

Lemma 4.2. *For any quasi-regular semi-perfectoid $\mathbb{Z}_p^{cyc}$-algebra R, there's a commutative diagram*

$$\begin{array}{ccc} T_p(R^\times) & \xrightarrow{-l_q} & \hat{\Delta}_R^{\varphi=\hat{\xi}} \\ \downarrow & & \downarrow \simeq \\ \pi_2(K(R;\mathbb{Z}_p)) & \xrightarrow{Ctr} & \pi_2(TC(R;\mathbb{Z}_p)) \end{array}$$

We're now in a position to state and prove our main theorem.

Theorem 4.3. *For any quasi-regular semi-perfectoid ring R, the map*

$$Hom_{(R)_{qsyn}}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \to Hom_{DM(R)}(M_\Delta(\mu_{p^\infty}), M_\Delta(\mathbb{Q}_p/\mathbb{Z}_p))$$

is a bijection.

Proof. Both the target and domain satisfy quasi-syntomic descent in R. So we can prove the assertion after passing to a quasi-syntomic cover, and thus by Andre's lemma we can assume that R is a $\mathbb{Z}_p^{cyc}$-algebra. Hence, we are reduced to showing that

$$l_q : T_p(R^\times) \to \Delta_R^{\varphi=\hat{\xi}}$$

is a bijection. Since the composite

$$T_p(R^\times) \to \hat{\Delta}_R^{\varphi=\hat{\xi}}$$

identifies with the cyclotomic trace, we can appeal to work of Clausen-Mathew-Morrow to assert that it's a bijection. So it suffices to show that

$$\Delta_R^{\varphi=\hat{\xi}} \xrightarrow{i} \hat{\Delta}_R^{\varphi=\hat{\xi}}$$

is injective. Notice that by definition of the Nygaard filtration, the Frobenius factors through a map $\psi : \hat{\Delta}_R \to \Delta_R$.

Now, for any $x \in \Delta_R^{\varphi=\hat{\xi}}$, if $i(x) = 0$, then

$$0 = \psi(0) = \psi(i(x)) = \varphi(x) = \hat{\xi}x.$$

But $\hat{\xi}$ is a non-zero divisor (by definition of prism), and thus $x = 0$. So i is injective, and we declare victory.

□

We'll now sketch the proof of the computation of the cyclotomic trace. The full proof is beyond the scope of these notes.

(sketch of lemma). The only fact that we need about the cyclotomic trace is that it lifts the Dennis trace, which is a map

$$Dtr : K(R; \mathbb{Z}_p) \to THH(R; \mathbb{Z}_p).$$

Bhatt, Morrow, and Scholze identify $\pi_{2i}(THH(R; \mathbb{Z}_p))$ with the i-th graded piece of the Nygaard filtration on (completed) prismatic cohomology. Furthermore, since R is quasi-regular semi-perfectoid, $A_{inf}(R) \xrightarrow{\theta} R$ is a surjection, so if J is the kernel, the HKR theorem implies we get an identification

$$\pi_2(THH(R; \mathbb{Z}_p)) \simeq J/J^2.$$

So we can summarize all the maps and identification that are playing a role in one big diagram:

$$
\begin{array}{ccccccc}
T_p(R^\times) & \longrightarrow & \pi_2(K(R;\mathbb{Z}_p)) & \xrightarrow{Ctr} & \pi_2(TC(R;\mathbb{Z}_p)) & \xrightarrow{\simeq} & \hat{\Delta}_R^{\varphi=\hat{\xi}} \\
 & & & \searrow^{Dtr} & \downarrow{\scriptstyle can} & & \downarrow \\
 & & & & \pi_2(THH(R;\mathbb{Z}_p)) & \xrightarrow{\simeq} & \mathcal{N}^{\geq 1}\hat{\Delta}_R/\mathcal{N}^{\geq 2}\hat{\Delta}_R \\
 & & & & \downarrow{\scriptstyle \simeq} & \nearrow & \\
 & & & & J/J^2 & &
\end{array}
$$

The proof now proceeds in essentially three steps. The first step is to reduce to a case where the natural projection

$$\hat{\Delta}_R^{\varphi=\hat{\xi}} \to \mathcal{N}^{\geq 1}\hat{\Delta}_R/\mathcal{N}^{\geq 2}\hat{\Delta}_R$$

is an injection. The reduction goes as follows. An element $x \in T_p(R^\times)$ is just a compatible system of p^n-th roots of unity. So since all the maps in this diagram are natural in R, we can pass to the universal case

$$R' := \mathbb{Z}_p \left\langle x^{\frac{1}{p^\infty}} \right\rangle /(x-1).$$

It turns out that the associated prism $(\Delta_{R'}, \hat{\xi})$ satisfies the property that $(p, \hat{\xi})$ is a regular sequence. For any prism of this form, the reduction map

$$A^{\varphi=\hat{\xi}} \to \mathcal{N}^{\geq 1}A/\mathcal{N}^{\geq 2}A$$

is injective. This takes some work, but it is a true fact.

Once we've made this reduction, it suffices to show that $-l_q = Ctr$ modulo $\mathcal{N}^{\geq 2}\hat{\Delta}_R$.

The second step is then to identify the q-logarithm modulo $\mathcal{N}^{\geq 2}\hat{\Delta}_R$. Since all the q-divided powers $\gamma_{n,q}$ lie in the n-th stage of the filtration, the logarithm assumes the form

$$l_q(x) = [x] - 1 \ \mathrm{mod} \mathcal{N}^{\geq 2}\hat{\Delta}_R.$$

In particular, the negative logarithm is given by

$$-l_q(x) = [x^{-1}] - 1 \ \mathrm{mod} \mathcal{N}^{\geq 2}\hat{\Delta}_R.$$

The final step utilize the first step once again to reduce to computing the Dennis trace rather than the cyclotomic trace, which turns out to be something we can get our hands on. Here we make use of the identification $\pi_2(THH(R;\mathbb{Z}_p)) \simeq I/I^2$ and compute the composite

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p)) \xrightarrow{Dtr} \pi_2(THH(R;\mathbb{Z}_p)) \simeq J/J^2$$

to be

$$x \to [x^{-1}] - 1.$$

The main point of this computation is a computation of the p-completed Hurewicz map

$$\pi_2(BG_p^\wedge) \xrightarrow{h} \pi_2((\mathbb{Z}[BG])_p^\wedge)$$

for any abelian group G. We then apply this computation to

$$T_p(R^\times) = \pi_2((BR^\times)_p^\wedge) \to \pi_2(\mathbb{Z}[BR^\times]_p^\wedge) \to \pi_2(HH(R; \mathbb{Z}_p)).$$

Of course, there's some minor book-keeping needed to make sure that the map

$$J/J^2 \to \mathcal{N}^{\geq 1}\hat{\Delta}_R/\mathcal{N}^{\geq 2}\hat{\Delta}_R$$

coming from the two different identification of $\pi_2(THH(R; \mathbb{Z}_p))$ does what you hope it does, but it does.

□